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MATRIX TRANSFORMATIONS IN SOME SEQUENCE SPACES

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Abstract: In this paper, we represent some sequence spaces and give the characterization of $(l(p), l_\infty)$, $(l(p), c)$, (w_p, c) and $(c_0(p), c_0(q))$.

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Introduction

Let X, Y be two nonempty subsets of the space of all complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers $a_{nk}(n, k = 1, 2, ...)$. For every $x = (x_k) \in X$ and every integer n we write

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. We say that $A \in (X, Y)$ if and only if $Ax \in Y$ when ever $x \in X$.
If $p_k > 0$ and sup$p_k < \infty$, we define (see Maddox [1])

$$l(p) = \left\{x : \sum k|x_k|^{pk} < \infty\right\}$$

$$c(p) = \left\{x : |x_k - 1|^{pk} \to 0 \text{ for some } 1\right\}$$

$$c_0(p) = \left\{x : |x_k|^{pk} \to 0\right\}$$

$$l_\infty(p) = \left\{x : sup|x_k|^{pk} < \infty\right\}$$

$$w(p) = \left\{ x : 1/n \sum_{k=1}^{n} |x_k - 1|^{pk} \to 0 \text{ for some } 1 \right\}$$

If $p_k = p$ for all k, then $l_\infty(p), c_0(p) = c(p)$
$l(p) = l_p$ and $w(p) = w_p$
l_∞, c and c_0 are respectively, the Banach spaces of bounded, convergent and null sequences.

The purpose of this paper is to characterize the matrices in the classes $(l(p), l_\infty)$, (l(p),c),$(w(p), c)$ and $(c_0(p), c_0(q))$.

The following notations are used through out for all integers $n \geq 1$, we write

$$Ax = (A_n(x))$$

If $A_n(x) = \sum_k a_{nk}x_k$ converges for each n.
For any two complex numbers a and b and $B > 0$
We write

$$|ab| \leq B(|a|^q B^{-a} + |b|^p) \tag{1}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
For all integers $n, m \geq 1$ and $N > 1$, we write

$$C(n, N.m, s) = \sum_{k=m}^{s} |a_{nk}|^{qk} N^{-qk}$$

where $1 \leq s \leq \infty$ and $\frac{1}{p_k} + \frac{1}{q_k} = 1$.
We put $C(n, N, 1, \infty) = C(n, N)$ and $C(N) = sup\, C(n, N)$ for every integer $N > 1$.
Theorem 1: Let $1 \leq p_k < sup\ p_k = H < \infty$ for every k. Then $A \in (l(p), l_\infty)$. If there is an integer $B > 1$ such that $C(B) < \infty$
Proof: Sufficiency

Let $x \in l(p)$ using inequality (1) we see that

$$|A_n(x)| \leq B(C(n, B)) + g^H(x) < B(C(B)) + g^H(x)$$

where $g^H(x) = \sum_k |x_k|^{pk}$.
Necessity

Let $A \in (l(p), l_\infty)$ but $C(N) = \infty$ for every integer $N > 1$. Then $\sum_k a_{nk}x_k$ converges for every n and every $x \in l(p)$ and each A_n defined by $A_n(x) = \sum a_n x_k$ is an element of $l * (p)$. Since $l(p)$ is complete and since $sup\ |A_n(x)| < \infty$ on $l(p)$

by uniform boundedness principal there exist a number M independent of n and x a number $\delta < 1$ such that

$$|A_n(x)| \leq M \tag{2}$$

For every $x \in s[0, \delta]$ and every n, where $s[0, \delta]$ denotes the closed sphere in $l(p)$ with center at the origin $0 = (0, 0, ...)$ and radius δ. Now choose an integer $Q > l$ such that

$$Q\delta^H > G$$

By our supposition we have $C(Q) = \infty$ and so two cases are possible:
Either $C(n, Q) < \infty$ for every $n \geq 1$ or there exists $n \geq 1$ such that $C(n, Q) = \infty$. In the first case there exists $n \geq 1$ such that $C(n, Q) > 2$ and there exists $k_0 > 1$ such that

$$C(n, Q, K_0 + 1, \infty) < 1$$

Hence

$$C(n, Q, l, k_0) > 1$$

In the second case we may choose $k_0 > 1$ such that $C(n, Q, l, k_0) > 1$, so that in either case there exists $n \geq 1$ and $k_0 \geq 1$ such that

$$S = C(n, Q, l, k_0) > 1 \tag{3}$$

Define a sequence $x = (x_k)$ as follows

$$x_k = \delta^{pk/H} (sgn\ a_{nk}) |a_{nk}|^{qk-1} S^{-1} Q^{-qk/pk} \text{ for } 1 \leq k \leq k_0,$$

$$= 0 \text{ for } \ k > k_0$$

It is easy to see that $g(x) \leq \delta$ and that

$$|A_n(x)| = s^{-1} Q \sum |a_{nk}|^{qk} Q^{-qk} \delta^{H/pk} \geq Q\delta^H > G$$

Which contradicts (2) and completes the proof.
Theorem 2:
Let $p = p_k$ be as in theorem 1, then $A \in (l(p), c)$ iff

(i) The condition of theorem 1 hold

(ii) $\lim_{n \to \infty} a_{nk} = a_k$ for each k.

Proof
Let $l \leq p_k \leq H < \infty$ for every k, since $e_k = \{\delta_{nk}\} \in l(p)$.
The necessity of (ii) follows
For the sufficiency observe that for every integer $m \geq 1$ and every n we have

$$C(n, B, 1, m) \leq C(B) \leq \infty$$

Hence

$$\lim_{m\to\infty} \lim_{n\to\infty} C(n, B, l, m) \leq C(B)$$

i.e.

$$\sum_k |a_k|^{qk} B^{-qk} \leq C(B)$$

Therefore the series $\sum a_k x_k$ and $\sum a_{nk} x_k$.
Converges for every n and every $x \in l(p)$. For each $x \in l(p)$ we can choose an integer $m \geq 1$ such that

$$\sum_{k=m+1}^{\infty} |x_k|^{pk} < 1$$

By using the inequality (1) it is easy to check that

$$\sum_{k=m+1}^{\infty} |a_{nk} - a_k||x_k| \leq 2B(2C+1) \quad \left(\sum_{k=m+1}^{\infty} |x_k|^{pk}\right)^{\frac{l}{H}}$$

It follows that

$$\lim_{n\to\infty} \sum_k a_{nk} x_k = \sum_k a_k x^k$$

This completes the proof.
Theorem 3:
Let $0 < p < 1$. then $A \in (w_p, c)$ if and only if

(i) $\lim_{n\to\infty} a_{nk} = a_k$ (k fixed)

(ii) $M(A) = sup \sum 2^{r/p} A_r^l(n) < \infty$, $nr = 0$, where $A_r^l(n) = max|a_{nk}|$ for each n, the maximum value is taken for r, k such that $2^r \leq k < 2^{r+1}$

Proof: Sufficiency
Suppose that the condition hold, since

$$M(A) = sup \sum 2^{r/p} A_r^l(n) < \infty, \quad nr = 0$$

where $A_r^l(n) = max|a_{nk}|$ for each n, the maximum is taken for k such that r, $2^r \leq k < 2^{r+1}$

$$\sum_{k=1}^{\infty}|a_{nk}x_k| = \sum_{r=0}^{\infty}\sum_{r}|a_{nk}x_k|$$

$$\leq \sum_{r=0}^{\infty}\left(\sum_{r}|a_{nk}x_k|^p\right)^{1/p}$$

$$\leq \sum_{r=0}^{\infty} A_r^l(n)2^{r/p}||x||^{1/p}$$

$$\leq M(A)||x||^{1/p} < \infty$$

Whenever $x \in w_p$ so the series $\sum_{k=1}^{\infty} a_{nk}x_k$ is absolutely convergent for each n.

Let $1/p + 1/q = 1, \;\; 0 < p < 1$, so that $q < 0$. Then for any positive integer 's' and for any $m \geq 2^s$ we have for all n,

$$\sum_{k=m}^{\infty}|a_{nk}| \leq \sum_{r=s}^{\infty}\sum_{r}|a_{nk}| \leq \sum_{r=s}^{\infty}A_r^l(n)2^r \leq M(A)2^{s/q}$$

Since $q < 0$ it follows that $\sum_{k=1}|a_{nk}|$ is uniformly convergent in n then this together with (i) implies that

$$\lim_{n\to\infty}\sum_{k} a_{nk} = \sum_{k} a_k \tag{4}$$

Now take $x \in w_p$ and suppose

$$1/n\sum_{k=1}^{n}|x_k - l|^p \to 0(n \to \infty)$$

Hence we write

$$\sum_{k} a_{nk}x_k = \sum_{k} a_k x_k + l\sum_{k}(a_{nk} - a_k) + \sum(a_{nk} - a_k)(x_k - l) \tag{5}$$

and from (i) and (ii) imply

$$\sum_{r=0}^{\infty} 2^{r/p} max|a_k| \leq M(A)$$

Hence $\sum_k |a_k x_k| < \infty$ and the last sum in (5) has limit zero as $n \to \infty$.
Thus by (4) we now have

$$\sum_k a_{nk} x_k \to \sum_k a_k x_k \quad \text{as } n \to \infty, \quad \text{for every x in} \quad w_p$$

This proves the sufficiency when $0 < p < 1$

Necessity

The necessity of (i) is trivial and we now prove the necessity of (ii).
Suppose that $A_n(x) = \sum_k a_{nk} x_k$ exists for each $n \geq 1$ whenever $x \in w_p$. Then for each n and each $r \geq 0$ the functional $f_{rn}(x) = \sum_r a_{nk} x_k$ are in the dual space w_p, they are trivially linear and continuous since

$$|f_{rn}(x)| \leq A_r^l(n) 2^{r/p} ||x||^{1/p}$$

It follows from Banach-Steinhaus theorem that for each n,

$$\lim_s \sum_{r=0}^{s} f_{rn}(x) = A_n x \quad \text{is in } w_p$$

Hence

$$|A_n(x)| \leq ||A_n|| \ ||x||^{1/p} \tag{6}$$

for each n we take any integer $s > 0$ and define $x \in w_p$ by $x_k = 0$ for $k \geq 2^{s+1}$,

$$x_{N(r)} = 2^{r/p} sgn\ a_n; N(r), x_k = 0 (k \neq N(r)) \quad \text{for} \quad 0 \leq r \leq s,$$

where $N(r)$ is such that $|a_n; N(r)| = max|a_{nk}|$
By (6) we get

$$\sum_{r=0}^{s} 2^{r/p} A_r^l(n) \leq ||A_n||$$

Hence for each n,

$$\sum_{r=0}^{\infty} 2^{r/p} A_r^l(n) \leq ||A_n|| < \infty \tag{7}$$

Now the argument used in the sufficiency to prove that the series defining $A_n(x)$ was absolutely convergent gives

$$|A_n(x)| \leq \sum_{r=0}^{\infty} 2^{r/p} A_r^l(n) ||x||^{1/p}$$

So that

$$||A_n|| \leq \sum_{r=0}^{\infty} 2^{r/p} A_r^l(n) \tag{8}$$

From (7) and (8) imply

$$||A_n|| = \sum_{r=0}^{\infty} 2^{r/p} A_r^l(n)$$

By Banach-Steinhaus theorem, the existence of lim $A_n(x)$ on w_p implies that

$$\sup ||A_n|| = \sup \sum_{r=0}^{\infty} 2^{r/p} A_r^l(n) < \infty$$

which is (ii). This complete the proof.

Theorem 4:

Let p be any positive sequence and q be a bounded sequence. Then a matrix $A = (a_{nk}) \in (c_0(p), c_0(q))$ iff

(i) $|a_{nk}|^{qn} \to 0$ ($n \to \infty$, each k)

(ii) $\lim\limits_N \limsup\limits_n \left(\sum\limits_k |a_{nk}| N^{-1/pk} \right)^{qn} = 0$

Proof

From (ii) the existence of a positive integer M such that

$$\sup \sum_{nk} |a_{nk}| M^{-1/pk} < \infty \tag{9}$$

Let us take $H = \sup q_k$ and observe that

$$|a_k + b_k|^{qk} \leq c(|a_k|^{qk} + |b_k|^{qk})$$

where $c = \max(1, 2^{H-1})$. For the sufficiency we take $\in > 0$ and let $x \in c_0(p)$. Since $c_0(p)$ is a (proper) subset of l_∞ we have $||x|| = \sup |x_k| < \infty$.
By (ii) there exist N such that

$$\sup \left(\sum_{nk} |a_{nk} N^{-1/pk}| \right)^{qn} < \in \tag{10}$$

Also there exist r such that $|x_k| < N^{-1/pk}$ for all $k > r$. Hence

$$|A_n(x)|^{qn} \leq c\left[||x||^{qn}\left(\sum_{k=1}|a_{nk}|^{qn/H}\right)\right]^H + \left[\sum_k |a_{nk}|N^{-1/p}\right]^{qn}$$

Taking lim sup in this last inequality we see that since $q \in l_\infty$ (i) and (10) implies $Ax \in c_0(q)$

Necessity:

The necessity of (i) is obtained by taking $x = c_k$ we prove the necessity of (ii).

We may suppose since $c_0(q) = c_0(q/H)$, that sup $q_k \leq 1$

Now $[c_0(p), c_0(q)][c_0(p), l_\infty]$ and observe that $A \in [c_0(p), l_\infty]$ iff (9) holds for some integer M.

If $p \in l_\infty$ then the necessity of the result is readily shown by an application of the uniform boundedness principle.

Let us denote the lim sup in (ii) by B_N, so that the sequence $(B_M, B_{M+1}, ...)$ is decreasing the limit in (ii) certainly exists; suppose if possible that it is equal to 3a, where $a > 0$, then there exist $n(1) > 1$ such that

$$\sum_{k>k(1)} |a_{n(1)k}|(M+1)^{-1/pk} > (2a)^{1/qn(1)}$$

Hence by (9) there exists $k(1) > 1$ such that

$$\sum_{k>k(1)} |a_{n(1)k}|(M+1)^{-1/pk} < (a/4)^{1/qn(1)}$$

Define

$$x_k = (sgna_{n(1)k})(M+1)^{-1/pk} \text{ for } 1 \leq k \leq k(1)$$

Suppose that $n(1) < n(2) < ... < n(i)$ and $k(1) < k(2) < ... < k(i)$ has been chosen and that x_k has been defined for $k \leq k(i)$ choose $n(i+1) > n(i)$ such that for $n = n(i+1)$

$$\sum_{ki} |a_{nk}|(M+i+1)^{-1/pk} > (2a)^{1/qn}$$

and

$$\sum_{k=1} |a_{nk}|^{qn} < a/4$$

Next; we choose $k(i+1) > k(i)$ such that for $n = n(i+1)$

$$\sum_{k>k(i+1)} |a_{nk}|(M+i+1)^{-1/pk} < (a/4)^{1/qn}$$

define
$$x_k = (sgn\ a_{n(i+1)k})(M+i+1)^{-1/pk} \ \text{ for } \ k(i) < k \leq k(i+1)$$
so that $x \in c_0(p)$, now for $n = n(i+1)$ write the sum for $a_{nk}x_k$ over $k(i) < k \leq k(i+1)$
as
$$\sum - \sum_i - \sum_{k>k(i+1)}$$
then,
$$|\sum_{1+k(i)} a_{nk}x_k|^{qn} > 2a - a/4 - a/4 = 3a/2 \tag{11}$$
By (11)
$$\begin{aligned} |A_n(x)|^{qn} &> 3a/2 - a/4 - \Big(\sum |a_{nk}x_k|\Big)^{qn} \\ &\geq 3a/2 - a/4 - a/4 \\ &= a \end{aligned} \tag{12}$$
Since $|x_k < (M+i+1)^{-1/pk}|$ for $k > k(i+1)$. But (12) is contrary to $Ax \in c_0(q)$. Hence (ii) is necessary.

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